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DISCRETE SYMMETRIES AND THE COMPLEX STRUCTURE OF CALABI-YAU MANIFOLDS

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ABSTRACT

We show how the discrete symmetries, which may be present after Calabi-Yau compactification for specific choices of the complex structure, extend to the $h_{2,1}$ moduli - the scalar fields whose vacuum expectation values determine the complex structure. This allows us to determine much about the coupling of the moduli and hence the energetically favoured complex structure. The discrete symmetry transformation properties of the moduli are worked out in detail for a three generation Calabi-Yau model and it is shown how minimization of the effective potential involving these fields selects the complex structure which leaves unbroken a set of discrete symmetries. The phenomenological implications of these symmetries are briefly discussed.

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The complex structure of a Calabi-Yau manifold K is determined by N complex parameters $a_{i=1...N}$ where N is given by $h_{2,1}$ the Hodge number counting the dimension of the $\bar{\partial}$ cohomology group $H^{(2,1)}(K)$. These complex parameters determine the magnitude of the Yukawa couplings of the effective four-dimensional gauge theory and for special values of the a_i this theory will have discrete symmetries [1,2] in addition to the gauge symmetries of the theory.

The parameters a_i should be interpreted [3] as vacuum expectation values (vevs) of the scalar components of chiral superfields which we denote by A_i. At tree level the components of A_i are massless and so the associated vevs, a_i, are undetermined. However, once one includes the effects of supersymmetry breaking, the scalar components of the A_i may have a non-trivial potential which determines their vevs and hence the complex structure of the manifold. If one can determine the couplings of the A_i the potential will be determined and, in principle, the vevs and complex structure can then be uniquely fixed.

In this paper we will be able to determine some relations amongst the couplings which follow from the (spontaneously broken) discrete symmetries of the manifold relevant for <u>arbitrary</u> complex structure. In this way we will be able to show, in a particularly interesting example, how the complex structure which leaves <u>unbroken</u> some discrete symmetries is favoured and we will briefly discuss possible phenomenological implications following from such symmetries, for example proton stability.

Let us start with what is known about the couplings of the A_i [3]. Since they are gauge singlet chiral superfields their couplings are solely determined by the superpotential. The A_i do not participate in any renormalizable Yukawa coupling to gauge non-singlet fermions V, U in the 27 or $\overline{27}$ representations respectively*) for a coupling $A_i V^k U_k$ would mean that a vev for A_i would cause the fields V^k and U_k to get a mass, contradicting the fact that the number of massless superfields transforming as the 27 or $\overline{27}$ are determined by topological invariants in the Hodge numbers. However, the A^i do have unrenormalizable Yukawa couplings to the fields transforming as the 27 because we know that the renormalizable Yukawa couplings of such fields depend on the complex structure of K, i.e., on the vevs of A_i. Indeed, for specific manifolds, the dependence of the Yukawa couplings on the complex structure is known [4,5] and this immediately translates to a knowledge of what non-renormalizable terms involving the A_i, trilinear in the superfields V^k, exist

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*) We assume a standard embedding of the gauge connection with gauge group $E_8 \times E_6$. Here we will only be interested in the visible sector with gauge group E_6.

in the superpotential. Unfortunately we are not able at present to extend this complete analysis beyond terms trilinear in the gauge non-singlet superfields so we turn here to a discussion of the discrete symmetry properties of the A_i which will be useful in relating such terms. A Calabi-Yau manifold of the complete intersection type is specified by the product of several CP_n spaces whose co-ordinates are restricted to lie on surfaces defined by polynomial equations involving these co-ordinates. For example, the only known Calabi-Yau space leading to just three generations is formed as the product of two CP_3 spaces whose co-ordinates x_i, y_i, $i=0,\ldots,3$ obey the defining equations [6]

$$\begin{aligned} &\sum a_{ijk}\, x_i x_j x_k = 0 \\ &\sum b_{ijk}\, y_i y_j y_k = 0 \\ &\sum c_{ij}\, x_i y_j = 0 \end{aligned} \tag{1}$$

This gives a manifold R_0 with Euler characteristic -18 corresponding to 9 generations. To reduce this number to three we form the manifold $R=R_0/G_0$ where G is the Z_3 freely acting group defined by

$$(x_0,x_1,x_2,x_3)\otimes(y_0,y_1,y_2,y_3) \xrightarrow[g\in G]{g} (x_0,\alpha^2 x_1,\alpha x_2,\alpha x_3)\otimes(y_0,\alpha y_1,\alpha^2 y_2,\alpha^2 y_3) \tag{2}$$

In this case the (complex) coefficients a_{ijk}, b_{ijk}, c_{ij} must be restricted so that Eq. (1) is invariant under the transformation of Eq. (2). Further we may always use a linear transformation amongst the homogeneous co-ordinates of $CP_3\times CP_3$ to put the defining equations in the form

$$\begin{aligned} &\sum x_i^3 + a_1 x_0 x_1 x_2 + a_2 x_0 x_1 x_3 = 0 \\ &\sum_i y_i^3 + b_1 y_0 y_1 y_2 + b_2 y_0 y_1 y_3 = 0 \\ &x_0 y_0 + c_1 x_1 y_1 + c_2 x_2 y_2 + c_3 x_3 y_3 + c_4 x_2 y_3 + c_5 x_3 y_2 = 0 \end{aligned} \tag{3}$$

The nine-independent parameters a_i, b_i, c_i correspond to the $h_{2,1} = 9$ independent deformations of R's complex structure [2]. Each of these parameters should be identified as the vev of a complex scalar component of a chiral superfield (denoted by A_i, B_i, C_i here).

We will use this example to show how the discrete symmetries of a manifold may be immediately determined; the analysis generalizes trivially to any manifold of the algebraic type. We first note that for the particular complex structure given by

$$\begin{aligned} &a_i = b_i = c_4 = c_5 = 0 \\ &c_1 = 1 \quad ; \quad c_2 = c_3 = c \end{aligned} \tag{4}$$

there is an 18 element non-Abelian discrete symmetry group, V, which leaves Eq. (3) invariant [2,7]. It is generated by

$$\begin{aligned} A:&\quad (x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3) \longrightarrow (\alpha x_0, \alpha x_1, x_2, x_3, \alpha^2 y_0, \alpha^2 y_1, y_2, y_3) \\ B:&\quad (x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3) \longrightarrow (x_0, x_1, \alpha x_2, x_3, y_0, y_1, \alpha^2 y_2, y_3) \\ C:&\quad (x_2, x_3, y_2, y_3) \longrightarrow (x_3, x_2, y_3, y_2) \end{aligned} \tag{5}$$

(The redundant generator A proves to be useful in later calculations.) In addition there is a $Z_2 \otimes Z_2$ symmetry of R generated by D and P where*)

$$\begin{aligned} D:&\quad (x_i, y_i) \longrightarrow (y_i, x_i) \\ P:&\quad (x_0, x_1, y_0, y_1) \longrightarrow (x_1, x_0, y_1, y_0) \end{aligned} \tag{6}$$

The symmetries A, B, C, D, P are exact symmetries of the manifold for the particular complex structure given in Eq. (4). However, note that since the complex coefficients of Eq. (4) are the vevs of the scalar components of the fields A_i, B_i, and C_i we may consider Eq. (3) as one in which a_i, b_i and c_i are the scalars components of A_i, B_i and C_i and not just their vevs. Thus we may immediately determine the transformation properties of A_i, B_i, and C_i under the discrete symmetries which leave Eq. (3) invariant. These are given in Table 1+). We see from Table 1 that when the

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*) These, can survive flux breaking when paired with a particular E_6 element. See Ref. [7].

+) Since the transformation properties of the C_i under P are somewhat awkward it is sometimes convenient to transform to different fields C_i' as given in Table 2.

vevs of these fields are given by Eq. (4) the discrete symmetries are unbroken, but if any of the vevs deviate from the form of Eq. (4) some of the discrete symmetries will be spontaneously broken.

How are these vevs fixed? As discussed above the moduli may not appear in the superpotential on their own, otherwise varying the complex structure will change the number of massless modes in contradiction with the topological information that there are $h_{2,1}$ such massless modes. Once supersymmetry is broken this is no longer true and the potential may have a non-trivial dependence on the moduli. It is easy to identify the dominant terms in models in which there is a large intermediate scale (I.S.) for non-renormalizable terms in the superpotential will then lead to large masses for the moduli through their coupling to the fields with the large I.S. These terms may be expected to dominate over radiative effects in which the moduli acquire masses via one or more loops with intermediate states either the massless modes of the theory or the massive excitations for in both cases the contribution will at most be $O[(\lambda^2/4\pi^2)m_{3/2}^2]$ where λ is some coupling.

To illustrate how the complex structure will be determined we turn again to the three generations Calabi-Yau example. After supersymmetry breaking it was shown in Ref. [7] that it may be energetically favourable to develop large vevs along specific D-flat directions with components of 27 and $\overline{27}$ acquiring the large vevs. For example, one such direction had (in the notation of Ref. [7]*)) non-zero vevs for $[\lambda_{1-}]_{(3,2)}$ and $[\bar{\lambda}_5,\bar{\lambda}_6]_{(3,2)}$. The leading dimension term in the superpotential which can lead to a non-vanishing contribution to the potential energy occurs at $O(27^2\ \overline{27}^2/M_C)$ where M_C is the compactification scale. Using the discrete symmetries we find the only troublesome term has the form

$$P_3 \propto \lambda_{1-}\lambda_{8-}\ f_2(\bar{\lambda}_5,\bar{\lambda}_6)/M_C \tag{7}$$

This could give a non-vanishing F term $F_{[\lambda_{8-}]_{(3.2)}}$ for the non-zero vevs considered above, but the relative vevs of the $\bar{\lambda}_5$ and $\bar{\lambda}_6$ will adjust to cancel in this F term. Thus the potential is F-flat in this direction beyond this order.

Now let us include the terms of leading dimension which may fix the vevs of the moduli. We first consider the fields A_1, A_2, B_1, B_2. From Table 1 and the transformation properties of the gauge non-singlet fields given in Ref. [2], we find the terms allowed by the discrete symmetries

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*) In what follows the subscripts ± refer to C eigenstates and $f_r(A,B)$ is a homogeneous function of degree r in the fields A and B. The subscripts (i,j) label the $SU(3)_L\times SU(3)_R$ indices in the decomposition E_6 $SU(3)_C\times SU(3)_L\times SU(3)_R$.

$$
\begin{aligned}
P_a &\propto \lambda_{1-}^2 \, f_2^a(\bar{\lambda}_5, \bar{\lambda}_6)\,(A_1+A_2)/M_c^2 \\
P_b &\propto \lambda_{1+}\lambda_{1-} \, f_2^b(\bar{\lambda}_5, \bar{\lambda}_6)\,(A_1-A_2)/M_c^2 \\
P_c &\propto \lambda_{3-}\lambda_{1-} \, f_2^c(\bar{\lambda}_5, \bar{\lambda}_6)\,(B_1+B_2)/M_c^2 \\
P_d &\propto \lambda_{3+}\lambda_{1-} \, f_2^d(\bar{\lambda}_5, \bar{\lambda}_6)\,(B_1-B_2)/M_c^2
\end{aligned}
\tag{8}
$$

From Eq. (8) and the non-zero vevs assumed above we see that the terms in the potential coming from the F terms $F_{[\lambda_1-](3,2)}$, $F_{[\lambda_1+](3,2)}$, $F_{[\lambda_3-](3,2)}$, and $F_{[\lambda_3+](3,2)}$ give positive masses squared to the scalar components of (A_1+A_2), (A_1-A_2), (B_1+B_2) and (B_1-B_2) respectively, thus causing them to have vanishing vevs. Comparison with Table 1 shows that this is sufficient to ensure that the A symmetry is exact, i.e., by minimizing the effective potential the complex structure is chosen which leads to an unbroken discrete symmetry. We may try to extend this analysis to the moduli C_i to see whether the maximally symmetric choice of Eq. (4) is obtained by minimization of the effective potential. In this case the allowed terms are

$$
\begin{aligned}
P_e &\propto \lambda_{8+}\lambda_{1-} \, f_2^e(\bar{\lambda}_5, \bar{\lambda}_6)\left[(C_2'-C_3'),\,(C_4'-C_5')\right]/M_c^2 \\
P_f &\propto \lambda_{5+}\lambda_{1-} \, f_2^f(\bar{\lambda}_5, \bar{\lambda}_6)\,(C_4'-C_5')/M_c^2 \\
P_g &\propto \lambda_{5-}\lambda_{1-} \, f_2^g(\bar{\lambda}_5, \bar{\lambda}_6)\,(C_4'+C_5')/M_c^2
\end{aligned}
\tag{9}
$$

which would cause $(C_2'-C'_3)$, C_4', C_5' to have vanishing vevs in order to set $F_{[\lambda_8+](3,2)}$, $F_{[\lambda_5+](3,2)}$ and $F_{[\lambda_5-](3,2)}$ to zero. From Table 2 we see that this is sufficient to ensure that the discrete symmetries generated by B, C and D are exact.

Thus we have seen in this simple case how supersymmetry breaking, manifested by large vevs for chiral superfields, may choose the complex structure which leaves a large set of discrete symmetries unbroken. Comparison with Eq. (4) shows that in our example this set of symmetries is almost the maximum set - with our choice of non-vanishing vevs. C_1' is not fixed at this order. In a realistic theory other non-vanishing vevs occur which determine C_1' too.

Note that the vev of the combination $(C_2'+C_3')$, which is left undetermined in Eq. (4), is not determined by terms of this order. Thus, although there is a term proportional to $\lambda_1 \text{-} \lambda_8 \text{-} F_2(\bar\lambda_5,\bar\lambda_6)\ (C_2'+C_3')$, this does not fix $(C_2'+C_3')$ because $F_{[\lambda_8 -](3,2)}$ coming from this term and Eq. (7) is minimized by adjusting the relative vevs of $\bar\lambda_5$ and $\bar\lambda_6$, and this may be done for arbitrary $(C_2'+C_3')$.

In superstring models the discrete symmetries play a very important phenomenological role. In particular they are needed to suppress unwanted processes such as proton decay, and to give structure to the fermion mass matrices [2,7,8]. Proton decay in particular is sensitive to any small derivations from exact symmetry [9,10] so it is important to determine the accuracy to which these symmetries hold. The stabilizing terms discussed above are quadratic in the moduli and any higher dimension term linear in the moduli will cause a small vev to develop. The dominant

source of such a term occurs if there is an F term with both constant and linear terms in the moduli of interest. In the model considered above a constant term occurs through the term $\lambda_{1-}^3\ f_3(\bar\lambda_5,\bar\lambda_6)$ which, with P_a of Eq. (8), leads to a vev for (A_1+A_2) given by

$$\langle (A_1+A_2)\rangle = O(\lambda_{1-}\bar\lambda_{5,6} M_C) \tag{10}$$

This will spontaneously break the A, B and D symmetries of Table 1 allowing terms forbidden by these symmetries to occur in the effective low-energy superpotential at $O(\langle A_1+A_2\rangle/M_C \equiv \langle\lambda_{1-}\rangle\langle\bar\lambda_{5,6}\rangle/M_C^2)$. For $\langle\lambda_{1-}\rangle$, $\langle\bar\lambda_{5,6}\rangle \leqslant O(10^{10}$ GeV) this will not induce, for example, proton decay at an unacceptable rate via dimension four operators. However, even for larger values of vev there are symmetries which may be used to prevent proton decay which are not broken so strongly. For example the C symmetry of Table 1 (which, in fact, was the one used in Ref. [7] to inhibit proton decay) is not broken by the the vev of Eq. (10). A vev for a field such as (A_1-A_2) which violates C does not occur by the above mechanism because there are no F terms linear in (A_1-A_2) which also have constant pieces. In this case a vev for (A_1-A_2) will only occur through the soft supersymmetry breaking term proportional to $[P]_A$, the A term of the superpotential. However, superstring models are thought to give rise to an effective theory in which such terms are negligible, so we expect a

negligible vev for (A_1-A_2) and hence an exact C symmetry suitable for inhibiting proton decay, etc.

In this letter we have shown how the discrete symmetry properties of the moduli may be determined for Calabi-Yau manifolds of the complete intersection type. Using these symmetries we may limit the form of the effective potential involving these moduli and hence obtain information about their vevs and the resulting complex structure. In the three generation example studied here we found that, after supersymmetry breaking, the vevs of the moduli were fixed and the complex structure was that which left a large set of discrete symmetries unbroken. We expect this to be a general feature of models with a large scale of vacuum expectation value for scalar fields for then there will be stabilizing terms for the moduli equivalent to those of Eqs. (8) and (9). The discrete symmetries emerging may play an important role in determining the phenomenology of the low-energy theory and many of them are likely to be unbroken to the precision needed if the symmetry is used to suppress dimension-four contributions to proton decay. In this way the phenomenological problems envisaged in Refs. [9] and [10] for Calabi-Yau models are elegantly avoided.

	A	B	C	D	P
A_1	α	α	$\to A_2$	$\to B_1$	1
A_2	α	1	$\to A_1$	$\to B_2$	1
B_1	α^2	α^2	$\to B_2$	$\to A_1$	1
B_2	α^2	1	$\to B_1$	$\to A_2$	1
C_1	1	1	1	1	$\to 1/C_1$
C_2	1	1	$\to C_3$	1	$\to C_2/C_1$
C_3	1	1	$\to C_2$	1	$\to C_3/C_1$
C_4	1	α	$\to C_5$	$\to C_5$	$\to C_4/C_1$
C_5	1	α^2	$\to C_4$	$\to C_4$	$\to C_5/C_1$

TABLE 1

Discrete symmetry properties of the $h_{2,1}$ moduli for the three generation Calabi-Yau manifold.

	A	B	C	D	P
C'_1	1	1	1	1	-1
C'_2	1	1	$\to C'_3$	1	1
C'_3	1	1	$\to C'_2$	1	1
C'_4	1	α^2	$\to C'_5$	$\to C'_5$	1
C'_5	1	α	$\to C'_4$	$\to C'_4$	1

TABLE 2

Discrete symmetries properties for the fields $C'_1=\ln C_1$, $C'_2=C_2^2/C_1$, $C'_3=C_3^2/C_1$, $C'_4=C_4^2/C_1$, $C'_5=C_5^2/C_1$.

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